FINITENESS OF NONZERO DEGREE MAPS BETWEEN THREE-MANIFOLDS

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ABSTRACT. In this paper, we prove that every orientable closed 3-manifold dominates at most finitely many homeomorphically distinct irreducible non-geometric 3-manifolds. Moreover, for any integer $d > 0$, every orientable closed 3-manifold d-dominates only finitely many homeomorphically distinct 3-manifolds.

1. INTRODUCTION

Let M, N be two orientable closed 3-manifolds. For an integer $d > 0$, we say that M *d-dominates* N if there is a map $f : M \to N$ of degree d up to sign. We say M *dominates* N if M d-dominates N for some integer $d > 0$. In this paper, we prove the following result:

Theorem 1.1. *Every orientable closed* 3*-manifold dominates at most finitely many homeomorphically distinct irreducible non-geometric* 3*-manifolds.*

In [BRW], Michel Boileau, Hyam Rubinstein and Shicheng Wang asked the question:

Question 1.2. Does every orientable closed 3-manifold dominate at most finitely many irreducible 3-manifolds supporting none of the geometries $\mathbb{S}^3$, $\widetilde{\mathrm{SL}}_2$, or Nil?

Note that any orientable closed 3-manifold supporting the geometry $\mathbb{S}^3$, $\widetilde{\mathrm{SL}}_2$, or Nil dominates infinitely many homeomorphically distinct 3-manifolds supporting the same geometry. Combined with known results for geometric targets, Theorem 1.1 completes a positive answer to Question 1.2, (cf. Section 2 for details). With a little extra effort, we can also deduce the following corollary, which in particular answers an earlier question of Yongwu Rong [Ki, Problem 3.100]:

Corollary 1.3. *For any integer $d > 0$, every orientable closed* 3*-manifold d-dominates only finitely many homeomorphically distinct* 3*-manifolds.*

Rong's original question asked only about 1-dominations. By around 2002, many partial results had been proved, including a complete affirmative answer to Rong's question in the geometric (target) case, contributed from different people ([So], [WZ], etc.). Here we refer the reader to an earlier survey [Wa] of Shicheng Wang. Those studies clarified the right expectation about finiteness of targets under dominations, eventually formulated as Question 1.2. We refer the readers to the introduction of [BRW] for more recent results on this topic.

The main technique of proving Theorem 1.1 is the presentation length estimation, as was used in [AL]. However, [AL] was concerned about knot complements, so the complexity of gluings was reduced to the complexity of geometric pieces using the desatellite trick. That trick does not work in general. As we shall see in Section 2, the finiteness of gluings

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becomes an essential issue to address in our current context. To illustrate the idea, let us assume at this point that M is an orientable closed 3-manifold that dominates an orientable closed 3-manifold N obtained from gluing two one-cusped hyperbolic 3-manifolds along the toral boundaries. Pulling straight f with respect to the hyperbolic geometry on both of the pieces, the area of $f(M^{(2)})$ should be at least the area in the neighborhood of the cutting torus $T \subset N$, namely, the sum of its areas within the (Margulis) cusps. When the gluing is complicated enough, there is at most one slope α on T which is short (i.e. no longer than some given bound imposed by M) on both sides. Pretend that $f(M^{(2)}) \cap T$ could be homotoped in T into a regular neighborhood of α, then $f : M \to N$ factors through the drilling $N - \alpha$ homotopically, so it cannot have nonzero degree. This provides an *a priori* upper bound on the complexity of gluings under the domination assumption. Similar as in [AL, Theorem 3.2], such factorization is not real in general, but it works in certain 'homological sense'. This is satisfactory enough for our applications. When Seifert fibered pieces are involved, the bound of the local complexity of gluings on each JSJ torus is not enough to determine the target N up to finitely many possibilities, so one should also count the total complexity of gluings on all the boundary components of a Seifert fibered piece. However, the spirit is similar to the local case. Finally, we remark that a significant shortcut in our treatment is Lemma 5.7, which uses a Poincaré-Lefschetz duality argument as a substitution of the factorization argument.

In Section 2, we give a brief review of the background and reduce the proof of Theorem 1.1 to bounding the complexity of gluings. In Section 3, we introduce the notion of *distortion* measuring the complexity of gluings. In Section 4, we show that any bound on the (primary average) distortion yields finiteness of gluings. In Section 5, we prove Theorem 1.1 by bounding the distortion of the targets. In Section 6, we prove Corollary 1.3 by generalizing previously known arguments for geometric target cases. In Section 7, we point out some further directions for studies in the future.

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2. Background

In this section, we review some known results, and explain how to reduce our problem to the finiteness of gluings. For standard terminology and facts of 3-manifold topology, cf. [Ja], and for the Geometrization Theorem, cf. [Th1], [MF].

Suppose N is an orientable closed irreducible 3-manifold. The Geometrization Theorem asserts that there is a canonical geometric decomposition cutting N along a minimal (possibly empty) finite collection of essential tori or Klein-bottles, unique up to isotopy, into compact *geometric pieces*, namely each supporting one of the eight 3-dimensional geometries of finite volume. This may be regarded as a graph-of-spaces decomposition of N, cf. Subsection 3.1.

When N is not itself geometric, all the pieces are either atoroidal, supporting the $\mathbb{H}^3$-geometry, or Seifert-fibered, supporting the $\mathbb{H}^2 \times \mathbb{E}^1$-geometry. When N is geometric, it is either atoroidal, supporting the $\mathbb{H}^3$-geometry, or Seifert-fibered, supporting one of the six geometries $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2$, $\mathbb{E}^3$, Nil, $\mathbb{S}^2 \times \mathbb{E}^1$ or $\mathbb{S}^3$, or otherwise, supporting the Sol-geometry. The geometry of the Seifert-fibered case can be determined according to the sign of the Euler characteristic $\chi \in \mathbf{Q}$ of the base-orbifold and whether the Euler number $e \in \mathbf{Q}$ of the fiberation vanishes.

The geometric decomposition of N usually coincides with the Jaco-Shalen-Johanson (JSJ) decomposition if one regards a compact regular neighborhood of each cutting Klein-bottle as a JSJ piece. The only exception occurs when N is Sol-geometric, and in this case, N has a nontrivial JSJ decomposition cutting N along an essential torus into either one *orientable thickened-torus* (i.e. the trivial interval-bundle over a torus) or two *orientable thickened-Klein-bottles* (i.e. the interval-bundle over a Klein-bottle whose bundle space is orientable).

In [BRW], Boileau, Rubinstein and Wang bounded the allowable JSJ pieces in an irreducible 3-manifold N dominated by an orientable closed 3-manifold M:

Theorem 2.1 ([BRW, Theorem 1.1])**.** *Let M be an orientable closed* 3*-manifold. Then there is a finite collection of orientable compact irreducible atoroidal or Seifert-fibered* 3*-manifolds with (possibly empty) incompressible tori boundary, such that for any orientable closed irreducible* 3*-manifold N which supports none of the geometries $\mathbb{S}^3$, $\widetilde{\mathrm{SL}}_2$ or* Nil*, if N is dominated by M, then any JSJ piece of N is homeomorphic to one of these* 3*-manifolds.*

Moreover, the finiteness of Sol-geometric targets under domination has been proved in [BBW]:

Theorem 2.2 ([BBW, Corollary 3.6])**.** *Every orientable closed* 3*-manifold dominates at most finitely many distinct* Sol*-geometric* 3*-manifolds.*

Therefore, to give a positive answer to Question 1.2, one still needs to bound the number of allowable targets that are non-geometric. This motivates Theorem 1.1.

Now suppose M is an orientable closed 3-manifold, and N is an irreducible non-Seifert-fibered 3-manifold dominated by M. It has been pointed out in [BRW, Lemma 4.2] that the Kneser-Haken number $h(M)$ (i.e. the maximal possible number of components of essential subsurfaces) of M bounds that of N, so the number of geometric pieces of N is at most $h(M) + 1$, and the number of cutting tori and Klein-bottles of N is at most $h(M)$. Thus there are at most finitely many allowable isomorphism types of the underlying graph of the geometric decomposition of N. As N is non-geometric, the allowable homeomorphism types of geometric pieces have been bounded already, so our goal is to bound the allowable ways to glue these pieces up.

3. Distortion of 3-manifolds

In this section, we introduce a geometric notion called the primary average distortion which measures the obstruction for an orientable closed irreducible 3-manifold to being geometric. For convenience, we prefer to define this for gluings, and it naturally implies the definition for 3-manifolds by the geometric decomposition.

3.1. **Gluings of geometric pieces.** Let N be an orientable closed irreducible 3-manifold. The geometric decomposition splits N as a graph-of-spaces, where each vertex corresponds to a geometric piece, and each edge corresponds to a cutting torus or Klein-bottle, joining vertices corresponding to the adjacent pieces. Since the regular neighborhood of a cutting Klein-bottle in N has only one boundary component, its corresponding edge has only one end, and should be regarded as a 'semi-edge'.

We introduce the notion of gluings in terms of graphs-of-spaces. Throughout this paper, we shall use the term *graph* specially meaning a graph possibly with semi-vertices and semi-edges. Precisely, a graph in our sense is a finite CW 1-complex Λ with a (possibly empty) subset of loop-edges marked as *semi-edges*, and with a (possibly empty) subset

of vertices marked as *semi-vertices*. We shall refer to other vertices and edges as *entire-vertices* and *entire-edges*, respectively. A entire-edge has two *ends*, but a semi-edge has only one. The *valence* of a vertex v is the number of distinct ends adjacent to v. For a graph Λ, we denote its set of vertices as $\mathrm{Ver}(\Lambda)$, and its set of edges as $\mathrm{Edg}(\Lambda)$. The set of ends-of-edges $\widetilde{\mathrm{Edg}}(\Lambda)$ is a branched two-covering of $\mathrm{Edg}(\Lambda)$ singular over all the semi-edges. The covering transformation takes every end δ to its *opposite end* $\bar{\delta}$, of the same edge that δ belongs to. See Figure 1 for an illustration.

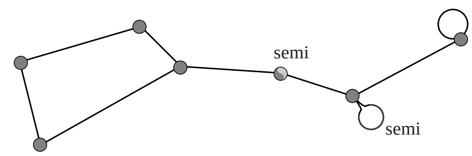

FIGURE 1. A graph with one semi-vertex and one semi-edge. It could be the graph of the geometric decomposition of a 3-manifold with seven geometric pieces, and eight cutting tori, and one cutting Klein-bottle. The semi-vertex corresponds to an $\mathbb{H}^2 \times \mathbb{E}^1$-geometric piece fibering over a non-orientable base-orbifold with two boundary components. The JSJ graph of this 3-manifold would have an edge instead of the semi-edge, which has a leaf vertex corresponding to an orientable thickened-Klein-bottle piece.

Definition 3.1. A *preglue graph-of-geometrics* is a finite graph Λ, together with an assignment of each vertex $v \in \mathrm{Ver}(\Lambda)$ to an oriented, compact, geometric 3-manifold J_v whose boundary consists of exactly n_v incompressible tori components, where n_v is the valence of v, and with an assignment of each end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$ adjacent to v to a distinct component T_δ of ∂J_v with the induced orientation. We require a semi-vertex be assigned to a J_v containing an embedded orientable thickened-Klein-bottle, and an entire vertex be assign to a J_v not as above. Let $\mathcal{J}$ be the disjoint union of all J_v's. We often ambiguously denote the preglue graph-of-geometrics as $(\Lambda, \mathcal{J})$.

Remark 3.2. It follows from the definition that a semi-vertex $v \in \mathrm{Ver}(\Lambda)$ can be characterized by that $J_v \subset \mathcal{J}$ is Seifert-fibered with a non-orientable base-orbifold.

Definition 3.3. A *gluing* of a preglue graph-of-geometrics $(\Lambda, \mathcal{J})$ is an assignment of each end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$ to an orientation-reversing homeomorphism $\phi_\delta : T_\delta \to T_{\bar{\delta}}$ between the tori assigned to δ and its the opposite end $\bar{\delta}$, up to isotopy, such that $\phi_{\bar{\delta}} = \phi_\delta^{-1}$ for any end-of-edge δ. Let:

$$\phi : \partial\mathcal{J} \to \partial\mathcal{J},$$

be the orientation-reversing involution defined by all ϕ_δ's. We often denote the gluing as ϕ, and denote the set of all gluings of $(\Lambda, \mathcal{J})$ as $\Phi(\Lambda, \mathcal{J})$.

A gluing ϕ is said to be *nondegenerate* if it does not match up ordinary-fibers in any pair of (possibly the same or via semi-edges) adjacent Seifert-fibered pieces.

For any gluing $\phi \in \Phi(\Lambda, \mathcal{J})$, one obtains a naturally associated orientable closed 3-manifold N_ϕ from $\mathcal{J}$ by identifying points in $\partial\mathcal{J}$ with their images under ϕ. It is clear that N_ϕ has the same geometric decomposition as prescribed by $(\Lambda, \mathcal{J})$ and ϕ if and only if ϕ is nondegenerate.

Let $\mathrm{Mod}(\partial\mathcal{J})$ be the special mapping class group of $\partial\mathcal{J}$, consisting of isotopy classes of component-preserving, orientation-preserving self-homeomorphisms of $\partial\mathcal{J}$. There is a natural (right) action of $\mathrm{Mod}(\partial\mathcal{J})$ on $\Phi(\Lambda,\mathcal{J})$. In fact, abusing the notations of isotopy classes and their representatives, for any $\tau\in\mathrm{Mod}(\partial\mathcal{J})$, and $\phi\in\Phi(\Lambda,\mathcal{J})$, one may define $\phi^\tau\in\Phi(\Lambda,\mathcal{J})$ to be:

$$\phi^\tau=\tau^{-1}\circ\phi\circ\tau,$$

namely, $(\phi^\tau)_\delta=\tau_{\bar\delta}^{-1}\circ\phi_\delta\circ\tau_\delta$ for each end-of-edge $\delta\in\widetilde{\mathrm{Edg}}(\Lambda)$, where $\tau_\delta\in\mathrm{Mod}(T_\delta)$ is the restriction of τ on the torus T_δ. It is straightforward to check that this is a well-defined, transitive action.

Definition 3.4. Two gluings $\phi,\phi'\in\Phi(\Lambda,\mathcal{J})$ are said to be *equivalent* if $\phi'=\phi^\tau$ for some $\tau\in\mathrm{Mod}(\partial\mathcal{J})$ that extends over $\mathcal{J}$ as a self-homeomorphism. Hence equivalent gluings yield homeomorphic 3-manifolds.

3.2. **Quadratic forms associated to gluings.** We introduce a positive-semidefinite quadratic form $\mathfrak{q}_\phi$ on $H_1(\partial\mathcal{J};\mathbf{R})$ for a gluing $\phi\in\Phi(\Lambda,\mathcal{J})$, which is nondegenerate if and only if so is ϕ.

Suppose J is an orientable compact geometric 3-manifold with nonempty tori boundary. Then ∂J consists of tori components, and J supports either the geometry $\mathbb{H}^3$ or $\mathbb{H}^2\times\mathbb{E}^1$. In both cases, there is a natural positive-semidefinite quadratic form:

$$\mathfrak{q}_J:\ H_1(\partial J;\mathbf{R})\to\mathbf{R},$$

on $H_1(\partial J;\mathbf{R})$ as follows.

If J is atoroidal, the interior of J has a unique complete hyperbolic metric ρ of finite volume by the Mostow rigidity, so we denote the $J_{\mathtt{geo}}=(\mathring{J},\rho)$. Then the induced conformal structures on the cusps endow $H_1(\partial J;\mathbf{R})$ with a canonical norm. Specifically, let $\epsilon>0$ be a sufficiently small Margulis number of $\mathbb{H}^3$, so that the compact ϵ-thick part $J^\epsilon_{\mathtt{geo}}$ of $J_{\mathtt{geo}}$ removes only horocusps of $J_{\mathtt{geo}}$. Then $\partial J^\epsilon_{\mathtt{geo}}$ is a disjoint union of tori $T^1\sqcup\cdots\sqcup T^q$ with induced Euclidean metrics, canonical up to rescaling for sufficiently small ϵ. We rescale the Euclidean metric on each T^j so that the shortest simple closed geodesic on T^j has length 1. This rescaled metric induces a Euclidean metric on the universal covering of T^j, and hence defines a canonical positive-definite quadratic form $\mathfrak{q}_{T^j}$ via the naturally induced inner product on $H_1(T^j;\mathbf{R})$. We define the positive-definite quadratic form $\mathfrak{q}_J$ on $H_1(\partial J;\mathbf{R})\cong H_1(T^1;\mathbf{R})\oplus\cdots\oplus H_1(T^q;\mathbf{R})$ to be the direct sum of the quadratic forms on its components.

If J is Seifert-fibered, as it supports the geometry $\mathbb{H}^2\times\mathbb{E}^1$, there is a canonical short exact sequence of groups:

$$1\longrightarrow\pi_1(S^1)\xrightarrow{\ i\ }\pi_1(J)\xrightarrow{\ p\ }\pi_1(O)\longrightarrow 1,$$

induced from the Seifert-fibration, where O is the hyperbolic base-orbifold. For any component $T\subset\partial J$, we regard $\pi_1(T)$ as a subgroup of $\pi_1(J)$, so we may first define for any $\zeta\in H_1(T;\mathbf{Z})\cong\pi_1(T)$ that $\mathfrak{q}_T(\zeta)$ equals the square of the divisibility of $p(\zeta)\in\pi_1(O)$ if $p(\zeta)$ is nontrivial, and equals zero if $p(\zeta)$ is trivial. This extends to a unique positive-semidefinite quadratic form $\mathfrak{q}_T$ on $H_1(T;\mathbf{R})$ which vanishes on the ordinary-fiber dimension. We define the positive-semidefinite quadratic $\mathfrak{q}_J$ on $H_1(\partial J;\mathbf{R})$ by summing up the quadratic forms on its components.

Definition 3.5. Suppose $(\Lambda,\mathcal{J})$ is a preglue graph-of-geometrics, and $\phi\in\Phi(\Lambda,\mathcal{J})$ is a gluing. For any end-of-edge $\delta\in\widetilde{\mathrm{Edg}}(\Lambda)$, let v, v' be the vertices adjacent to δ and its

opposite $\overline{\delta}$, respectively. For any $\zeta \in H_1(T_\delta; \mathbf{R})$, we define:

$$\mathfrak{q}_\phi(\zeta) = \mathfrak{q}_{J_v}(\zeta) + \mathfrak{q}_{J_{v'}}(\phi_\delta(\zeta)).$$

Note this is also well-defined when $\delta = \overline{\delta}$. We define the positive-semidefinite quadratic from $\mathfrak{q}_\phi$ on:

$$H_1(\partial\mathcal{J}; \mathbf{R}) = \bigoplus_{\delta \in \widetilde{\mathrm{Edg}}(\Lambda)} H_1(T_\delta; \mathbf{R}),$$

to be the direct sum of the quadratic forms on its components.

In other words, $\mathfrak{q}_\phi$ is the quadratic form pulled back from the graph of $\phi_* : H_1(\partial\mathcal{J}; \mathbf{R}) \to H_1(\partial\mathcal{J}; \mathbf{R})$, namely the image of $\mathrm{id} \oplus \phi_*$ in $(H_1(\partial\mathcal{J}; \mathbf{R}) \oplus H_1(\partial\mathcal{J}; \mathbf{R}),\ \mathfrak{q}_{\partial\mathcal{J}} \oplus \mathfrak{q}_{\partial\mathcal{J}})$. It is clear that $\mathfrak{q}_\phi$ is positive-definite if and only if ϕ is nondegenerate. Thus in this case, $H_1(\partial\mathcal{J}; \mathbf{R})$ is a Euclidean space with the induced inner product structure. We also remark that equivalent gluings induce the same quadratic form.

3.3. **Distortion of gluings.** Given a preglue graph-of-geometrics $(\Lambda, \mathcal{J})$, we first introduce average distortions of a gluing $\phi \in \Phi(\Lambda, \mathcal{J})$ at vertices and along edges of Λ. Roughly speaking, these measure the local complexity of a gluing, or more precisely, the local obstruction to extending the geometry across these objects.

Recall that for any free $\mathbf{Z}$-module V of finite rank $n \geq 0$, and a quadratic form $\mathfrak{q}$ on $V_{\mathbf{R}} = V \otimes_{\mathbf{Z}} \mathbf{R}$ over $\mathbf{R}$, the discriminant:

$$\Delta(V, \mathfrak{q}) \in \mathbf{R},$$

is the determinant of the associated bilinear form of $\mathfrak{q}$ over a (hence any) basis of V. When $\mathfrak{q}$ is positive-definite, it equals the square of the volume of the n-dimensional flat torus $V_{\mathbf{R}} / V$ with the Euclidean structure of $V_{\mathbf{R}}$ induced from $\mathfrak{q}$.

Definition 3.6. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a gluing, and let $e \in \mathrm{Edg}(\Lambda)$ be an (entire or semi) edge. We define the *average distortion* (or simply, the *distortion*) of ϕ along e as:

$$\mathscr{D}_e(\phi) = \Delta\left(H_1(T_\delta; \mathbf{Z}),\ \mathfrak{q}_\phi\right)^{\frac{1}{4}},$$

where δ is an end of e. Note the definition does not depend on the choice of the end.

Definition 3.7. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a gluing, and let $v \in \mathrm{Ver}(\Lambda)$ be a vertex of valence n_v. Suppose $n_v > 0$. If v is an entire-vertex, we define the *average distortion* (or simply, the *distortion*) of ϕ at v as:

$$\mathscr{D}_v(\phi) = \Delta\left(\partial_* H_2(J_v, \partial J_v; \mathbf{Z}),\ \mathfrak{q}_\phi\right)^{\frac{1}{2n_v}},$$

where $\partial_* H_2(J_v, \partial J_v; \mathbf{Z})$ denotes the image of $H_2(J_v, \partial J_v; \mathbf{Z})$ in $H_1(\partial\mathcal{J}; \mathbf{R})$ under the natural boundary homomorphism. If v is a semi-vertex, J_v is Seifert-fibered with a non-orientable base-orbifold. Let $\tilde{J}_v$ be the double covering of J_v corresponding to the centralizer of its ordinary-fiber, and let $\tilde{\mathfrak{q}}_\phi$ on $H_1(\partial\tilde{J}_v; \mathbf{R})$ be the direct sum of the quadratic forms on each component $H_1(\tilde{T}; \mathbf{R})$ pulled back from $\mathfrak{q}_\phi$, where $\tilde{T} \subset \partial\tilde{J}_v$. We define:

$$\mathscr{D}_v(\phi) = \Delta\left(\partial_* H_2(\tilde{J}_v, \partial\tilde{J}_v; \mathbf{Z}),\ \tilde{\mathfrak{q}}_\phi\right)^{\frac{1}{4n_v}}.$$

We also define $\mathscr{D}_v(\phi) = 0$ if $n_v = 0$.

Remark 3.8. Note the definition of average distortion along entire edges can be restated in a similar fashion if one takes a compact regular neighborhood $\mathcal{U}_e$ of T_e in place of the role of J_v above, because $\partial_* H_2(\mathcal{U}_e, \partial\mathcal{U}_e; \mathbf{Z}) \cong H_1(T_\delta)$ is a canonical isomorphism. One can also restate the definition of average distortion along semi-edges.

Definition 3.9. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a gluing of a preglue graph-of-geometrics $(\Lambda, \mathcal{J})$. We define the *primary average distortion* (or simply, the *primary distortion*) of ϕ as:

$$\mathscr{D}_\Lambda(\phi) = \max\{\mathscr{D}_v(\phi),\ \mathscr{D}_e(\phi) \,|\, v \in \mathrm{Ver}(\Lambda),\ e \in \mathrm{Edg}(\Lambda)\}.$$

For any orientable closed irreducible 3-manifold N, we may realize N by the natural nondegenerate gluing ϕ of the preglue graph-of-geometrics $(\Lambda, \mathcal{J})$ associated to its geometric decomposition, so the *primary distortion* of N is defined as:

$$\mathscr{D}(N) = \mathscr{D}_\Lambda(\phi).$$

This is clearly well-defined.

Remark 3.10. The adjectives 'average' and 'primary' suggest there could be other reasonable formulations of local and global distortions. For instance, at an entire-vertex, we actually think of the average distortion as the geometric mean of 'the distortions on the principal directions' on $\partial_* H_2(J_v, \partial J_v; \mathbf{R})$ with respect to the Euclidean norm defined by $\mathfrak{q}_\phi$ and the lattice $\partial_* H_2(J_v, \partial J_v; \mathbf{Z})$. The same idea applies to entire-edges and we think of average distortions for semi objects as the average distortions after passing to a natural double covering.

The distortion measures how far the 3-manifold is from being geometric:

Lemma 3.11. *For an orientable closed irreducible* 3*-manifold N, the primary distortion $\mathscr{D}(N)$ vanishes if and only if N is geometric.*

Proof. It follows immediately from the fact that geometric decompositions must not match up ordinary-fibers in adjacent Seifert-fibered pieces. In fact, as the associated gluing is nondegenerate if N is non-geometric, no average distortion vanishes along any edge. □

Passing to finite covering of the graph preserves the primary distortion:

Lemma 3.12. *If $\tilde{N}$ is a finite covering of an orientable closed irreducible* 3*-manifold N, so that every geometric piece $\tilde{J} \subset \tilde{N}$ covers is its underlying image $J \subset N$ either homeomorphically, or doubly corresponding to the ordinary-fiber centralizer (only if J is Seifert-fibered over a non-orientable base-oribifold), then $\mathscr{D}(\tilde{N}) = \mathscr{D}(N)$.*

Proof. This follows immediately from definition. In fact, the induced finite coverings of the graphs $\tilde{\Lambda} \to \Lambda$ (regarded as 'orbi-graphs') has the same index, and it preserves the average distortion at vertices and along edges. □

4. Distortion and gluings

In this section, we show a finiteness result that there are only finitely many homeomorphically distinct orientable closed irreducible 3-manifolds obtained from nondegenerate gluings of a nontrivial preglue graph-of-geometrics with bounded primary distortion. This is an immediate consequence of the following:

Proposition 4.1. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics. For any $C > 0$, there are at most finitely many distinct nondegenerate gluings $\phi \in \Phi(\Lambda, \mathcal{J})$ up to equivalence, such that $\mathscr{D}_\Lambda(\phi) < C$.*

We prove Proposition 4.1 in the rest of this section. Our strategy is as follows: using distortion along edges, we bound the allowable gluings up to fiber-shearings (see Definition 4.2 below); then using the distortion at Seifert-fibered vertices, we shall bound the allowable indices of fiber-shearings, and hence the allowable gluings up to equivalence. This will prove Proposition 4.1. Note distortion at atoroidal vertices is not used in our proof. As

we shall explain in Subsection 4.4, this is implied as the distortion at an atoroidal vertex is bounded in terms of the distortions along its adjacent edges.

4.1. **Fiber-shearings.** We introduce an operation called a fiber-shearing for a gluing $\phi \in \Phi(\Lambda, \mathcal{J})$. It corresponds to a surgery on an ordinary fiber of a Seifert-fibered piece of the associated 3-manifold N_ϕ, which preserves the base-orbifold.

Recall that for an oriented torus T and a slope $\gamma \subset T$, the (right-hand) *Dehn-twist* along γ is self-homeomorphism $D_\gamma \in \mathrm{Mod}(T)$ so that $D_\gamma(\zeta) = \zeta + \langle \zeta, \gamma \rangle \gamma$ for any slope ζ, where $\langle \cdot, \cdot \rangle$ denotes the intersection form. Note this does not depend on the direction of γ. For any integer k, a *k-times Dehn-twist* along γ is known as the k-times iteration D^k_γ.

Definition 4.2. Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics. We say $\tau \in \mathrm{Mod}(\partial \mathcal{J})$ is a *fiber-shearing* with respect to $(\Lambda, \mathcal{J})$ if for each end $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$ adjacent to a vertex v, $\tau_\delta \in \mathrm{Mod}(T_\delta)$ is either the identity, if J_v is atoroidal, or a k_δ-times Dehn-twist along the ordinary-fiber, where k_δ is an integer, if J_v is Seifert-fibered. The *index* of τ at a Seifert-fibered vertex v is the integer:

$$k_v(\tau) = \sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} k_\delta,$$

where $\widetilde{\mathrm{Edg}}(v)$ denotes the set of ends adjacent to v. For any gluing $\phi \in \Phi(\Lambda, \mathcal{J})$, the *fiber-shearing* of ϕ under τ is the gluing $\phi^\tau \in \Phi(\Lambda, \mathcal{J})$.

Note that the index is additive for products of fiber-shearings.

Lemma 4.3. *Fiber-shearings of the same index at all Seifert-fibered vertices yield equivalent gluings.*

Proof. It suffices to show that a fiber-shearing with zero index at all Seifert-fibered vertices does not change the equivalence class of a gluing. This follows immediately from the fact that for any pair of boundary tori T, T' in a Seifert-fibered piece J, there is a properly embedded annulus A bounding a pair of ordinary-fibers, one on each component. As the annulus A is two-sided when J is oriented, there is a well-defined Dehn-twist on J along this annulus, restricting to a right-hand Dehn-twist on T and a left-hand Dehn-twist (i.e. the inverse of a right-hand Dehn-twist) on T'. □

4.2. **Distortion along edges.** In this subsection, we show that distortion along edges bounds nondegenerate gluings up to fiber-shearings. This follows from a general fact about twisted sum of positive semi-definite quadratic forms. Although we shall only apply the rank two case of Proposition 4.5 in our estimations, it might be worth pursuing a little more generality for certain independent interest.

We first mention an easy fact in linear algebra.

Lemma 4.4. *Let V be a free $\mathbf{Z}$-module of finite rank $n > 0$, and $\mathfrak{q}$ be a positive-definite quadratic form on $V_{\mathbf{R}} = V \otimes_{\mathbf{Z}} \mathbf{R}$. For any $C > 0$, and any integer $0 \leq k \leq n$, there are at most finitely many rank-k submodules W of V with the discriminant $\Delta(W, \mathfrak{q}) < C$.*

Proof. Fix a basis $e_1, \cdots, e_n$ of V. It suffices to prove for the Euclidean form $\mathfrak{q}_0$ induced by the fixed basis as an orthonormal basis, since the nondegeneracy ensures $\Delta(W, \mathfrak{q}_0) < \lambda \cdot \Delta(W, \mathfrak{q})$ for some $\lambda > 0$ depending only $\mathfrak{q}$. Note that rank-k submodules of V are in bijection with rank-1 submodules of $\wedge^k V$, represented by primitive elements $w \in \wedge^k V$ up to sign. As $\wedge^k V$ has a natural inner product with a standard orthonormal basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}$, for any $\pm w \in \wedge^k V$ representing W, the well-known Cauchy-Binet formula implies:

$$\Delta(W, \mathfrak{q}_0) = \|w\|^2,$$

where $\|\cdot\|$ is the norm induced from the inner product structure. As w is an integral linear combination of the basis vectors, there are at most finitely many primitive w's satisfying $\|w\| < C$. □

Let V be a free $\mathbf{Z}$-module of finite rank $n \geq 0$. The special linear group $\Gamma = \mathrm{SL}(V)$ acts naturally (from the right) on the space of quadratic forms on $V_{\mathbf{R}}$, namely, any $\tau \in \Gamma$ transforms a quadratic form $\mathfrak{q}$ into the composition $\mathfrak{q}\tau$. We write the stabilizer of $\mathfrak{q}$ in Γ as $\Gamma_{\mathfrak{q}}$. We say a quadratic form $\mathfrak{q}$ has *rational kernel* with respect to the lattice $V \subset V_{\mathbf{R}}$, if the kernel $U_{\mathbf{R}}$ of (the associated bilinear form of) $\mathfrak{q}$ in $V_{\mathbf{R}}$ intersects V in a lattice (i.e. a discrete cocompact subgroup) $U \subset U_{\mathbf{R}}$.

Proposition 4.5. *With notations above, let $\mathfrak{q}, \mathfrak{q}'$ be two positive-semidefinite quadratic forms on $V_{\mathbf{R}}$ over $\mathbf{R}$ with rational kernels with respect to V. Note that the value of $\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}')$ depends only on the double-coset $\Gamma_{\mathfrak{q}}\sigma\Gamma_{\mathfrak{q}'}$. Then for any $C > 0$, there are at most finitely many distinct double-cosets $\Gamma_{\mathfrak{q}}\sigma\Gamma_{\mathfrak{q}'}$ of Γ, such that the discriminant:*

$$0 < \Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') < C.$$

Proof. We denote the unit-balls of $\mathfrak{q}$ and $\mathfrak{q}'$ as B and B', respectively. The unit-ball B_σ of $\mathfrak{q}\sigma + \mathfrak{q}'$ is clearly contained in $\sigma^{-1}(B) \cap B'$. When $\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') > 0$, B_σ is compact, but B or B' may be noncompact if $\mathfrak{q}$ or $\mathfrak{q}'$ are degenerate. See Figure 2.

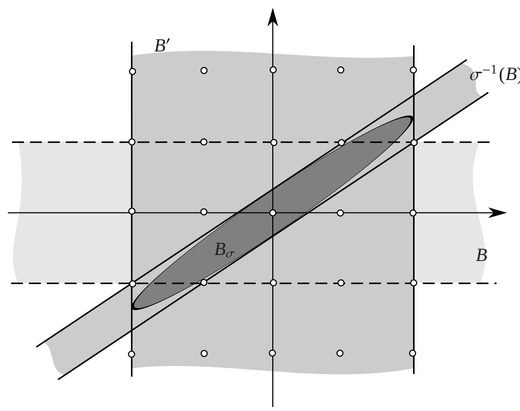

FIGURE 2. The unit balls, illustrated as $n = 2$ and $k = k' = 1$.

We claim that for any $C > 0$, there exists some compact subset:

$$K \subset V_{\mathbf{R}},$$

such that for any $\sigma \in \Gamma$ with $0 < \Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') < C$, there is some $\tau' \in \Gamma_{\mathfrak{q}'}$, such that the unit-ball $B_{\sigma\tau'}$ of $\mathfrak{q}(\sigma\tau') + \mathfrak{q}'$ is contained in K.

To prove this claim, we need to understand the action of $\Gamma_{\mathfrak{q}'}$. Let $U'_{\mathbf{R}}$ be the kernel of $\mathfrak{q}'$, of dimension k', and let $U' = U'_{\mathbf{R}} \cap V$ be the sublattice intersecting V. As $\mathfrak{q}'$ has rational kernel, U' also has rank k', and V splits as $U' \oplus L'$ for some sublattice L' of rank $n - k'$. Pick a basis $\xi'_1, \cdots, \xi'_{k'}$ of U' and a basis $\xi'_{k'+1}, \cdots, \xi'_n$ of L'. Hence they form a basis of V. Now $\Gamma_{\mathfrak{q}'}$ has a free abelian subgroup Π' generated by the 'elementary shearings' $\tau'_{ij} \in \Gamma$, defined for any $1 \leq i \leq k'$, and $k' + 1 \leq j \leq n$, by the identity on all the basis vectors except for:

$$\tau'_{ij}(\xi'_j) = \xi'_i + \xi'_j.$$

In particular, Π fixes the subspace $U'_{\mathbf{R}}$. Moreover, $\Gamma_{\mathfrak{q}'}$ has a natural subgroup isomorphic to $\mathrm{SL}(U')$, acting on the $U'_{\mathbf{R}}$ factor while fixing the $L'_{\mathbf{R}}$ factor. In fact, these two subgrops generate a finite-index normal subgroup of $\Gamma_{\mathfrak{q}'}$, which is a semidirect product $\Pi' \rtimes \mathrm{SL}(U')$.

We fix a reference Euclidean metric on $V_{\mathbf{R}}$ with the orthonormal basis $\xi'_1, \cdots, \xi'_n$, and denote the induced m-dimensional volume measure on any m-dimensional subspace as μ_m. It will be also conventient to make the convention that the zero-dimensional volume of the origin is one. The volume of B_σ is proportional to the reciprocal of the square root of $\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}')$, indeed:

$$\mu_n(B_\sigma) = \frac{\omega_n}{\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}')^{\frac{1}{2}}},$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of an n-dimensional Euclidean unit-ball. Thus the assumption $0 < \Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') < C$ is equivalent to:

$$\frac{\omega_n}{\sqrt{C}} < \mu_n(B_\sigma) < \infty.$$

Up to a composition by some τ' in $\mathrm{SL}(U') \leq \Gamma_{\mathfrak{q}'}$, we may first assume $\sigma^{-1}(B) \cap U'_{\mathbf{R}}$ is bounded within a uniform distance $D_1 > 0$ from the origin. In fact, for any $\sigma \in \Gamma$, we have:

$$\frac{\omega_n}{\sqrt{C}} < \mu_n(B_\sigma) \leq \mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}}) \cdot \mu_{n-k'}(B' \cap (U'_{\mathbf{R}})^\perp),$$

so $\mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}})$ is bounded below in terms of C. On the other hand,

$$\mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}}) = \frac{\omega_{k'}}{\Delta(U', \mathfrak{q}\sigma)^{\frac{1}{2}}} = \frac{\omega_{k'}}{\Delta(\sigma(U'), \mathfrak{q})^{\frac{1}{2}}},$$

and $\Delta(\sigma(U'), \mathfrak{q})$ further equals the discriminant of the embedded image $\overline{\sigma(U')}$ of $\sigma(U')$ in the quotient $V\,/\,U$, with respect to the induced nondegenerate quadratic form $\bar{\mathfrak{q}}$. Thus, the uniform lower bound of $\mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}})$ yields a uniform upper bound of $\Delta(\overline{\sigma(U')}, \bar{\mathfrak{q}})$. By Lemma 4.4, at most finitely many rank-k' submodules of $V\,/\,U$ are allowed to be the image $\overline{\sigma(U')}$. Furthermore, if two images $\overline{\sigma_0(U')}$ and $\overline{\sigma_1(U')}$ coincide, the identification pulls back to be an isomorphism τ' in $\mathrm{SL}(U')$, so that $\sigma_1 = \sigma_0\tau'$ restricted to U'. In other words, there are at most finitely many $\sigma^{-1}(B) \cap U'_{\mathbf{R}}$ up to compositions by elements of $\mathrm{SL}(U') \leq \Gamma_{\mathfrak{q}'}$. Hence they can be bounded uniformly within a uniform distance $D_1 > 0$ from the origin.

Now we also pick a splitting $V = U \oplus L$, and correspondingly pick a basis $\xi_1, \cdots, \xi_k$ of U and a basis $\xi_{k+1}, \cdots, \xi_n$ of L, in a similar fashion as before. For any $\sigma \in \Gamma$ with $\frac{\omega_n}{\sqrt{C}} < \mu_n(B_\sigma) < \infty$, we can find $k+1 \leq j_1 < j_2 < \cdots < j_h \leq n$, where $h = n - k - k'$, such that $U'_{\mathbf{R}}$ is transversal to the subspace:

$$\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}),$$

where $H_{\mathbf{R}}$ is spanned by $\xi_{j_1}, \cdots, \xi_{j_h}$.

Let σ be as above. Up to a composition by some τ' in $\Pi \leq \Gamma_{\mathfrak{q}'}$, we may further assume $\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}) \cap B'$ bounded within a uniform distance $D_2 > 0$ from the origin. In fact, we may find vectors:

$$\eta'_j = y_{1j}\,\xi'_1 + \cdots + y_{k'\,j}\,\xi'_{k'} + \xi'_j,$$

for each $k'+1 \leq j \leq n$, such that the η'_j together span $\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}})$. Note τ_{ij} fixes η'_t when $t \neq j$, and changes only the i-th coordinate of η'_j by $+1$. Thus, using $\sigma\tau'$ instead of σ for some $\tau' \in \Pi$, we may assume that $0 \leq y_{ij} < 1$ for all the y_{ij}'s above. Let $R > 0$ be sufficiently large, so that every point in $L'_{\mathbf{R}} \cap B'$ is bounded within in the radius R ball centered at the origin. Then every point in $\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}) \cap B'$ is bounded within the radius

$\sqrt{k'+1}R$ ball centered at the origin, so we take this radius as the uniform $D_2 > 0$. Note that for a different $\sigma \in \Gamma$ one may need to pick a different coordinate subspace $H_{\mathbf{R}} \leq L_{\mathbf{R}}$, (indeed, there could be up to $\binom{n-k}{h}$ choices), but the constant $D_2 > 0$ depends only on $\mathfrak{q}'$. Note also that this does not affect $\sigma^{-1}(B) \cap U'_{\mathbf{R}}$ which we have already taken care of.

Under the adjustment assumptions above, every vector $v \in B_\sigma$ can be written as $u' + w$, where $u' \in \sigma^{-1}(B) \cap U'_{\mathbf{R}}$, and $w \in \sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}) \cap B'$ for some appropriate $H_{\mathbf{R}}$. Thus, for any $\sigma \in \Gamma$ with $0 < \Delta(\mathfrak{q}\sigma + \mathfrak{q}') < C$, we have shown that there is some $\tau' \in \Gamma_{\mathfrak{q}'}$ so that $B_{\sigma\tau'}$ is bounded within the uniform radius $D_1 + D_2$ ball centered at the origin. Taking this uniform large ball as K, we have proved the claim.

To complete the proof of Proposition 4.5, observe that there is a uniform positive lower bound of the length of the short-axis of the ellipsoid B_σ, provided that B_σ is bounded within K. This is clear because the volume $\mu_n(B_\sigma)$ is at least $\frac{\omega_n}{\sqrt{C}}$. It follows that for all such σ's, $(\mathfrak{q}\sigma + \mathfrak{q}')(\xi'_t)$ is bounded by some uniform constant, for every $1 \leq t \leq n$. Suppose:

$$\sigma(\xi'_t) = x_{t1}\,\xi_1 + \cdots + x_{tn}\,\xi_n,$$

where $x_{t1}, \cdots, x_{tn}$ are integers. Because $\mathfrak{q}$ vanishes restricted to $U_{\mathbf{R}}$ and is nondegenerate restricted to $L_{\mathbf{R}}$, we obtain a uniform upper bound for every x_{jt}, where $k+1 \leq j \leq n$, and $1 \leq t \leq n$. Hence at most finitely many integers are allowed to be the the coefficients of the $\xi_{k+1}, \cdots, \xi_n$ components. Moreover, whenever two σ_0, σ_1 coincide on these coefficients, they differ only by a post-composition of some $\tau \in \Gamma$, which preserves U and induces the trivial action on $V\,/\,U$. Such a τ belongs to $\Gamma_{\mathfrak{q}}$, so $\Gamma_{\mathfrak{q}}\sigma_0 = \Gamma_{\mathfrak{q}}\sigma_1$.

To sum up, we have shown that for every $\sigma \in \Gamma$ with $0 < \Delta(\mathfrak{q}\sigma + \mathfrak{q}') < C$, every left-coset $\sigma\Gamma_{\mathfrak{q}'}$ contains a representative so that B_σ is bounded in some uniform compact set K, and that these representatives belong to at most finitely many distinct right-cosets $\Gamma_{\mathfrak{q}}\sigma$. This means there are at most finitely many distinct double-cosets $\Gamma_{\mathfrak{q}}\sigma\Gamma_{\mathfrak{q}'}$. ◻

Lemma 4.6. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics. For any $C > 0$, there are at most finitely many nondegenerate distinct gluings $\phi \in \Phi(\Lambda, \mathcal{J})$ up to fiber-shearings, such that $\mathscr{D}_e(\phi) < C$ for every edge $e \in \mathrm{Edg}(\Lambda)$.*

Proof. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a nondegenerate gluing satisfying the conclusion. For any end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, $\phi_\delta : T_\delta \to T_{\bar\delta}$ induces the quadratic form $\mathfrak{q}_\phi| = \mathfrak{q}_{J'}\phi_\delta + \mathfrak{q}_J$ on $H_1(T_\delta; \mathbf{R})$, where J, J' are the pieces containing T_δ, $T_{\bar\delta}$, respectively. Pick a reference gluing $\psi_\delta : T_\delta \to T_{\bar\delta}$, then $\phi_\delta = \psi_\delta\sigma$ for some $\sigma \in \mathrm{Mod}(T_\delta)$. Write $\mathfrak{q} = \mathfrak{q}_J$, and $\mathfrak{q}' = \mathfrak{q}_{J'}\psi_\delta$, and $\Gamma = \mathrm{Mod}(T_\delta)$, then $\mathfrak{q}_\phi$ on $H_1(T_\delta; \mathbf{R})$ equals $\mathfrak{q}\sigma + \mathfrak{q}'$ for some $\sigma \in \Gamma$. Clearly the stabilizer $\Gamma_{\mathfrak{q}}$ of $\mathfrak{q}$ in Γ is nontrivial only if J is Seifert-fibered, in which case $\Gamma_{\mathfrak{q}}$ is generated by a Dehn-twist along an ordinary-fiber on T_δ; and the stabilizer $\Gamma_{\mathfrak{q}'}$ is nontrivial only if J' is Seifert-fibered, in which case $\Gamma_{\mathfrak{q}'}$ is generated by a Dehn-twist along an ordinary-fiber on $T_{\bar\delta}$ pulled back on T_δ via ψ_δ. By the assumption and the definition of edge distortion, $\Delta(H_1(T_\delta; \mathbf{Z}), \mathfrak{q}\sigma + \mathfrak{q}') < C$. Moreover, $\Delta(H_1(T_\delta; \mathbf{Z}), \mathfrak{q}\sigma + \mathfrak{q}') > 0$ because ϕ is nondegenerate. Thus Proposition 4.5 implies that there are at most finitely many allowable types of ϕ_δ up to fiber-shearings. As $\phi : \partial\mathcal{J} \to \partial\mathcal{J}$ is defined by all the ϕ_δ's where $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, we conclude there are at most finitely many nondegenerate gluings ϕ up to fiber-shearings, which have edge distortions all bounded by C. ◻

4.3. **Distortion at Seifert-fibered vertices.** In this subsection, we show that distortion at Seifert-fibered vertices bounds nondegenerate fiber-shearings of a given gluing up to equivalence, and prove Proposition 4.1.

Lemma 4.7. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics, and $\phi \in \Phi(\Lambda, \mathcal{J})$ be a nondegenerate gluing. Suppose $v \in \mathrm{Ver}(\Lambda)$ is a Seifert-fibered vertex. Then for any $C > 0$, there exists some $K > 0$, depending on C and ϕ, such that whenever ϕ^τ is a fiber-shearing of ϕ with $\mathscr{D}_v(\phi^\tau) < C$, the fiber-shearing index $k_v(\tau)$ satisfies $|k_v(\tau)| < K$.*

Proof. There are two cases according to v being entire or semi.

Case 1. v is an entire-vertex, i.e. J_v has an orientable base-orbifold.

In this case, we pick consistent directions for all the fibers of J_v, and for any end-of-edge δ adjacent to v, let λ_δ be the directed slope on $T_\delta \subset \partial J_v$. Suppose the valence of v is $n_v > 0$. It is not hard to see that $\partial_* H_2(J_v, \partial J_v; \mathbf{Z}) < H_1(\partial J_v; \mathbf{R})$ has a rank-$(n_v - 1)$ submodule:

$$L_v \;=\; \left\{ \sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} l_\delta\,[\lambda_\delta] \,\Big|\, \sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} l_\delta = 0,\ \text{where } l_\delta \in \mathbf{Z} \right\},$$

and that there is an element:

$$[\mu_v] = \sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} [\mu_\delta] \in \partial_* H_2(J_v, \partial J_v; \mathbf{Z}),$$

such that for each $\delta \in \widetilde{\mathrm{Edg}}(v)$, $[\mu_\delta] \in H_1(T_\delta, \mathbf{Z})$ and the intersection number $\langle \mu_\delta, \lambda_\delta \rangle = m_v$ where $m_v > 0$ is the least common multiple of the orders of cone-points on the base-orbifold. Moreover,

$$\partial_* H_2(J_v, \partial J_v; \mathbf{Z}) \;=\; L_v \,\oplus\, \mathbf{Z} \cdot [\mu_v].$$

For simplicity, we write $\mathfrak{q}, \mathfrak{q}^\tau$ for $\mathfrak{q}_\phi, \mathfrak{q}_{\phi^\tau}$. Note that $\mathfrak{q}^\tau = \mathfrak{q}$ restricted to L_v.

We estimate the value of $\mathfrak{q}^\tau$ over the coset $[\mu_v] + L_v \otimes \mathbf{R}$ of $\partial_* H_2(J_v, J_{\partial v}; \mathbf{R})$. For any $[\xi] = \sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} l_\delta\,[\lambda_\delta] \in L_v \otimes \mathbf{R}$,

$$\mathfrak{q}^\tau\left([\mu_v] + [\xi]\right) = \sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathfrak{q}\left([\mu_\delta] + (l_\delta + m_v k_\delta)\,[\lambda_\delta]\right),$$

where $m_v > 0$ is as above, and k_δ is the Dehn-twist number on T_δ as in the definition of fiber-shearings. We have:

$$\sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} (l_\delta + m k_\delta) = m_v k_v(\tau),$$

so if $|k_v(\tau)| \geq K$, there must be one end $\delta^* \in \widetilde{\mathrm{Edg}}(v)$, such that $|l_{\delta^*} + m_v k_{\delta^*}| \;\geq\; K/n_v$. Thus:

$$\begin{aligned}
\mathfrak{q}^\tau\left([\mu_v] + [\xi]\right) \quad &\geq \quad \mathfrak{q}\left([\mu_{\delta^*}] + (l_{\delta^*} + m_v k_{\delta^*})\,[\lambda_{\delta^*}]\right) \\
&\geq \quad \frac{1}{2}\,\mathfrak{q}\left((l_{\delta^*} + m_v k_{\delta^*})[\lambda_{\delta^*}]\right) - \mathfrak{q}\left([\mu_{\delta^*}]\right) \\
&\geq \quad \frac{K^2 r_v}{2 n_v} - R_v
\end{aligned}$$

where $r_v = \min_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathfrak{q}([\lambda_\delta])$ and $R_v = \max_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathfrak{q}([\mu_\delta])$ are constants depending only on J_v and ϕ. Note $r_v > 0$ because ϕ is nondegenerate.

Now we have:

$$\mathscr{D}_v(\phi^\tau) = \left(\Delta(L_v, \mathfrak{q}) \cdot \inf_{[\xi] \in L_v \otimes \mathbf{R}} \{\mathfrak{q}([\mu_v] + [\xi])\}\right)^{\frac{1}{2n_v}} \geq \left(\Delta_{L_v} \cdot \Big(\frac{K^2 r_v}{2n_v} - R_v\Big)\right)^{\frac{1}{2n_v}},$$

where $\Delta_{L_v} = \Delta(L_v, \mathfrak{q}) > 0$ because ϕ is nondegenerate. In other words, if $\mathscr{D}_v(\phi^\tau) < C$, we obtain an upperbound $K > 0$ so that the absolute value of the fiber-shearing index $k_v(\tau)$ is bounded by K.

Case 2. v is a semi-vertex, i.e. J_v has a non-orientable base-orbifold.

In this case, let $\tilde{J}_v$ be the double covering of J_v corresponding to the centralizer of ordinary-fiber as in the definition of the vertex distortion. Then $\partial\tilde{J}_v$ is a trivial double covering of ∂J_v, and every fiber-shearing $\tau \in \mathrm{Mod}(\partial J_v)$ at v of index $k_v(\tau)$ lifts to a unique $\tilde{\tau} \in \mathrm{Mod}(\partial\tilde{J}_v)$ of index $2k_v(\tau)$. As now $\tilde{J}_v$ is Seifert-fibered over an orientable-orbifold, we reduce to the previous case, bounding the absolute value of $2k_v(\tau)$ by some K depending on C and ϕ. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. By Lemma 4.6, there are at most finitely many allowable types of gluings up to fiber-shearings. By Lemma 4.7, for each allowable fiber-shearing family $\{\phi^\tau\}$ as $\tau \in \mathrm{Mod}(\partial\mathcal{J})$ runs over all fiber-shearings where ϕ is a reference nondegenerate gluing, there are at most finitely many allowable indices of τ at any Seifert-fibered vertex. Hence by Lemma 4.3, there are at most finitely many distinct nondegenerated gluings up to equivalence with bounded primary distortion. □

4.4. **Distortion at atoroidal vertices: a remark.** The reader may have noticed that distortion at atoroidal vertices are not used in the proof of Proposition 4.1. The following lemma provides some reason behind.

Lemma 4.8. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics, and $v \in \mathrm{Ver}(\Lambda)$ be a vertex of valence n_v, corresponding to an atoroidal piece $J_v \subset \mathcal{J}$. Then for any gluing $\phi \in \Phi(\Lambda, \mathcal{J})$,*

$$\mathscr{D}_v(\phi) \leq C \cdot \left(\prod_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathscr{D}_{e(\delta)}(\phi) \right)^{\frac{2}{n_v}},$$

where $\widetilde{\mathrm{Edg}}(v)$ denotes the ends-of-edges adjacent to v, and $e(\delta)$ denotes the edge containing the end-of-edge δ, and $C > 0$ is some constant depending only on the topology of J_v.

Proof. For simplicity we rewrite J_v as J, and n_v as n. Write the submodule $\partial_* H_2(J, \partial J; \mathbf{Z})$ of $H_1(\partial J; \mathbf{Z})$ as W, and the subspace $\partial_* H_2(J, \partial J; \mathbf{R})$ of $H_1(\partial J; \mathbf{R})$ as $W_{\mathbf{R}}$. From the definition, we have $\mathfrak{q}_\phi \geq \mathfrak{q}_J$, both positive-definite on $H_1(\partial J; \mathbf{R})$, (cf. Subsection 3.2), so the unit-ball B_ϕ of $\mathfrak{q}_\phi$ is contained the (compact) unit-ball B_J of $\mathfrak{q}_J$. It suffices to show for some $C_0 > 0$ independent of ϕ,

$$\Delta(W, \mathfrak{q}_\phi) \leq C_0 \cdot \Delta(H_1(\partial J; \mathbf{Z}), \mathfrak{q}_\phi).$$

Picking a basis of $H_1(\partial J; \mathbf{Z})$ as an orthonormal basis, we fix a reference inner product of $H_1(\partial J; \mathbf{R})$. Denote the induced $2n$-dimensional volume measure as μ_{2n}, and denote the induced n-dimensional volume measure on $W_{\mathbf{R}}$ and on $W_{\mathbf{R}}^\perp$ as μ_n. It suffices to show for some $C_1 > 0$ independent of ϕ,

$$\mu_{2n}(B_\phi) \leq C_1 \cdot \mu_n(W_{\mathbf{R}} \cap B_\phi).$$

Note that:

$$\mu_{2n}(B_\phi) = \frac{\omega_{2n}}{\omega_n^2} \cdot \mu_n(W_{\mathbf{R}} \cap B_\phi) \cdot \mu_n(\bar{B}_\phi),$$

where ω_m denotes the volume of an m-dimensional Euclidean unit-ball, and $\bar{B}_\phi$ is the image of the orthogonal projection of B_ϕ to $W^\perp$. The last inequality follows immediately as:

$$\mu_n(\bar{B}_\phi) \leq \mu_n(\bar{B}_J),$$

where $\bar{B}_J$ is the image of the orthogonal projection of B_J to $W^\perp$. The right-hand side is finite, independent from ϕ. □

5. Domination onto non-geometric 3-manifolds

In this section, we bound the primary distortion of gluings under the assumption of domination, namely:

Proposition 5.1. *Suppose M is an orientable closed 3-manifold, and N_ϕ is an orientable closed irreducible 3-manifold obtained from a nondegenerate gluing $\phi \in \Phi(\Lambda, \mathcal{J})$ of a preglue graph-of-geometrics $(\Lambda, \mathcal{J})$. Then there exists some $C > 0$, such that if M dominates N_ϕ, then the primary distortion $\mathscr{D}_\Lambda(\phi) < C$.*

It is clear that Theorem 1.1 follows immediately from Propositions 4.1, 5.1, as equivalent gluings yield homeomorphic 3-manifolds by Definition 3.4.

We prove Proposition 5.1 in the rest of this section. In Subsection 5.1, we reduce the proof to the case when the underlying graph Λ is loopless and entire, and we show this case in Subsection 5.2.

5.1. Reduction to loopless entire graphs.

We say a graph Λ is *loopless* if it contains no loop edge. We say a graph is *entire* if there is no semi-edge or semi-vertex.

Lemma 5.2. *If Proposition 5.1 holds under the assumption that Λ is loopless and entire, it holds in general as well.*

Proof. The idea is that Λ, as an 'orbi-graph', has covering $\tilde{\Lambda}$ of index at most four which is loopless and entire. To be precise, suppose $f : M \to N_\phi$ is a nonzero degree map. We rewrite N_ϕ as N for simplicity. Take two copies X_0, X_1 of the compact 3-manifold obtained by cutting N along a maximal disjoint union of incompressible Klein-bottles, and glue each component of ∂X_0 to a unique component of ∂X_1 according to the gluing pattern of N. Then we obtain a double covering $\tilde{N}'$ of N, whose graph $\tilde{\Lambda}'$ is entire, (possibly disconnected if Λ is itself entire). Now cut $\tilde{N}'$ along the tori corresponding to the loop edges of $\tilde{\Lambda}'$, and glue two copies of the resulting compact 3-manifold up according to the gluing pattern of $\tilde{N}'$. Then we obtain a double covering $\tilde{N}''$ of $\tilde{N}'$, whose graph $\tilde{\Lambda}''$ is loopless and entire, (possibly disconnected if $\tilde{\Lambda}'$ is already loopless). Pick a connected component of $\tilde{N}''$, and rewrite as $\tilde{N}$. Thus $\tilde{N}$ covers N of index at most four, and has a loopless entire graph $\tilde{\Lambda}$. Moreover, $\mathscr{D}(\tilde{N}) = \mathscr{D}(N)$ by Lemma 3.12. However, $\tilde{N}$ is dominated by a (connected) covering $\tilde{M}$ of M with index at most four, so $\mathscr{D}(\tilde{N})$ is at most some $c(\tilde{M}) > 0$, where $c(\tilde{M})$ is a constant guaranteed by the assumption. Note there are only finitely many such $\tilde{M}$'s, since $\pi_1(M)$ is finitely generated. Let $C > 0$ be the maximum among all the $c(\tilde{M})$, as $\tilde{M}$ runs over all the coverings of M with index at most four. Thus $\mathscr{D}_\Lambda(\phi) = \mathscr{D}(N) < C$, as $\phi \in \Phi(\Lambda, \mathcal{J})$ is a nondegenerate gluing. □

5.2. The loopless entire graph case.

In this subsection, we prove Proposition 5.1 for the loopless entire graph case, namely:

Proposition 5.3. *Proposition 5.1 holds under the assumption that Λ is loopless and entire.*

We prove Proposition 5.3 in the rest of this subsection. In fact, we show that under the assumption of Proposition 5.3, the distortion $\mathscr{D}_v(\phi) < C$ at any vertex $v \in \mathrm{Ver}(\Lambda)$, where $C > 0$ depends only on the *triangulation number* $t(M)$ of M, i.e. the minimal possible number of tetrahedra in a triangulation of M; and similarly, $\mathscr{D}_e(\phi) < C$ for any edge $e \in \mathrm{Edg}(\Lambda)$. Our approach here is similar to the method used in [AL, Section 3], and these distortions can also be bounded using the presentation length of $\pi_1(M)$ ([AL, Definition 3.1]), if one prefers a group-theoretic point of view. However, to bypass unnecessary technicalities, we avoid formulating factorization results in this paper.

To simplify the notations, we rewrite N_ϕ and N in the rest of this subsection, and assume without loss of generality that N is not geometric. Let:

$$\mathcal{T} = \bigsqcup_{e \in \mathrm{Edg}(\Lambda)} T_e \subset N,$$

be the union of cutting tori of N in its geometric decomposition, and let:

$$\mathcal{U} = \bigsqcup_{e \in \mathrm{Edg}(\Lambda)} \mathcal{U}_e \subset N,$$

be a compact regular neighborhood of $\mathcal{T}$, cf. Figure 3. Note $\partial\mathcal{U}$ can be naturally identified as the disjoint union of tori T_δ's, where $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, and the complement in N of the interior of $\mathcal{U}$ can be naturally identified with the disjoint union of the geometric pieces $\mathcal{J}$. We make this identification throughout this subsection, so:

$$N = \mathcal{J} \cup_{\partial\mathcal{U}} \mathcal{U}.$$

Take a minimal triangulation of M, namely, a finite 3-dimensional simplicial complex structure on M with the fewest possible 3-simplices. We often denote $M^{(i)} \subset M$ for the i-skeleton of M, where $0 \leq i \leq 3$. As there are $t = t(M)$ tetrahedra, $M^{(2)}$ contains exactly $2t$ triangles. Let:

$$f : M \to N,$$

be a nonzero degree map from M to N as assumed. We may homotope f to be piecewise linear so that $f^{-1}(\mathcal{T}) \subset M$ becomes a normal surface of minimal complexity with respect to the triangulation of M, (i.e. minimizing the cardinality of $f^{-1}(\mathcal{T}) \cap M^{(1)}$), and that $f^{-1}(\mathcal{U}) \subset M$ is an interval bundle over $f^{-1}(\mathcal{T})$.

To further adjust f by homotopy, we need some geometry of N. Let $\epsilon_3 > 0$ be the Margulis constant of $\mathbb{H}^3$, so every $0 < \epsilon < \epsilon_3$ is a proper Margulis number of $\mathbb{H}^3$ (hence also of $\mathbb{H}^2$). For any $0 < \epsilon < \epsilon_3$, we may endow N with a Riemannian metric ρ_ϵ satisfying that for any $v \in \mathrm{Ver}(\Lambda)$: if J_v is atoroidal, then (J_v, ρ_ϵ) is isometric to the corresponding complete hyperbolic 3-manifold $J_v^{\mathtt{geo}}$ with open ϵ-thin horocusps removed; or if J_v is Seifert-fibered, (J_v, ρ_ϕ) is isometric to a corresponding complete $\mathbb{H}^2 \times \mathbb{E}^1$-geometric 3-manifold $J_v^{\mathtt{geo}}$ with open horizontal-ϵ-thin horocusps removed. Here by *horizontal* we mean with respect to the pseudo-metric pulled back from the metric on the hyperbolic base-orbifold, so for instance, a horizontal-ϵ-thin horocusp means the preimage in $J_v^{\mathtt{geo}}$ of a ϵ-thin horocusp in $\mathcal{O}^{\mathtt{geo}}$.

With the Riemanian metric ρ_ϵ on N, one may speak of the *area* for any piecewise-linearly immersed CW 2-complex $j : K \to N$, or for any integral (cellular) 2-chain of K. Specifically, note that for each hyperbolic J_v, there is an area measure on $f^{-1}(J_v) \cap K$ pulling back the hyperbolic area measure on J_v, and for each $\mathbb{H}^2 \times \mathbb{E}^1$-geometric J_v, there is an area measure on $f^{-1}(J_v) \cap K$ puling back the horizontal-area measure on J_v, (heuristically the area after projecting onto the base-orbifold, cf. [AL, Subsection 3.3] for details). Thus, the *area* of K with respect to f is known as the sum of the area measures of $K \cap f^{-1}(J_v)$ for all $v \in \mathrm{Ver}(\Lambda)$, denoted as $\mathrm{Area}(f(K))$; and the area of an integral 2-chain of K is the sum of the areas of its simplices weighted by the absolute values of their coefficients.

With the Riemannian metric ρ_ϵ on N, we 'pull straight' $f|_{M^{(2)}}$ within each J_v relative to ∂J_v, namely:

Lemma 5.4. *If $\epsilon > 0$ is sufficiently small, then the map $f : M \to N$ can be homotoped relative to $f^{-1}(\mathcal{U})$, so that $f(M^{(2)}) \cap \mathcal{J}$ is ruled on each component of the image of the 2-simplices of M, and that the area of $M^{(2)}$ is at most $2t\pi$, where t is the number of tetrahedra in the triangulation of M.*

Proof. To sketch the proof, pick a subdivision of the components of $M^{(2)} \setminus (f^{-1}(\mathcal{U}) \cup M^{(1)})$ into the fewest possible triangles. First homotope f relative to $f^{-1}(\mathcal{U})$, so that the image of the sides of these triangles becomes geodesic in their corresponding pieces. Then relatively homotope f further, so that the image of these triangles becomes ruled in their corresponding pieces. If we fix $f|_{M^{(0)}}$, as $\epsilon \to 0$, the image of these triangles converges to geodesic (possibly degenerate) triangles in hyperbolic pieces, and to horizonally-geodesic (possibly degenerate) triangles in $\mathbb{H}^2 \times \mathbb{E}^1$-geometric pieces (in the sense of being geodesic after projecting onto the base-orbifold). Moreover, for each 2-simplex of $M^{(2)}$, all except at most one triangle above contained in this 2-simplex becomes degenerate in the above sense, while the exceptional one has area at most π. Thus, for sufficiently small $\epsilon > 0$, the area of $M^{(2)}$ can be bounded by $2t\pi$ where $2t$ is the number of 2-simplices of $M^{(2)}$ with our notations. Using standard hyperbolic geometry estimations, it is not hard to make the arguments above rigorous, but we omit the details here for conciseness. □

Let $\epsilon_3 > 0$ denote the Margulis constant of $\mathbb{H}^3$. For each edge e adjacent to v, let:

$$\mathcal{W}_e \subset N,$$

be the union of $\mathcal{U}_e$ together with the compact ϵ_3-thin (or horizontal-ϵ_3-thin) horocusp neighborhoods of its adjacent pieces, (depending on e is entire or semi, there could be either two or one such horocusps). Possibly after an arbitrarily small shrinking of $\mathcal{W}_e$, we may assume the union of $\mathcal{W}_e$'s is still a compact regular neighborhood of $\mathcal{T}$, properly containing $\mathcal{U}$ whenever $\epsilon < \epsilon_3$; and we may also assume that $f^{-1}(\partial\mathcal{W}_e)$ intersects $M^{(2)}$ in general positions, i.e. that any 2-simplex of $M^{(2)}$ is transversal to $\partial\mathcal{W}_e$ under f. As Λ is a loopless graph, each $\mathcal{W}_e$ deformation-retracts to T_e, so there is a quadratic form on the subspace $\partial_* H_2(\mathcal{W}_e, \partial\mathcal{W}_e; \mathbf{R})$ of $H_1(\partial\mathcal{W}_e; \mathbf{R})$, naturally induced from $\mathfrak{q}_\phi$ on $H_1(T_\delta; \mathbf{R}) \oplus H_1(T_{\bar\delta}; \mathbf{R})$, where $\delta, \bar\delta$ are the two ends of e. Furthermore, for each $v \in \mathrm{Ver}(\Lambda)$, let:

$$\mathcal{W}_v \subset N,$$

be the union of J_v together with all the $\mathcal{W}_e$'s where e runs over edges adjacent to v. See Figure 3. As Λ is a loopless graph, each $\mathcal{W}_v$ deformation-retracts to J_v, so there is a quadratic form on the subspace $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$ of $H_1(\partial\mathcal{W}_v; \mathbf{R})$, naturally induced from $\mathfrak{q}_\phi$ on $H_1(\partial J_v; \mathbf{R})$.

These $\mathcal{W}_e$'s and $\mathcal{W}_v$'s are natural geometric objects associated with the geometric decomposition of N. The following isoperimetric comparison is the junction point between the geometric aspect and the quadratic forms associated to gluings:

Lemma 5.5. *For any vertex $v \in \mathrm{Ver}(\Lambda)$, if $j : (S, \partial S) \to (\mathcal{W}_v, \partial\mathcal{W}_v)$ is a properly piecewise-linearly immersed oriented compact surface, then:*

$$\mathrm{Area}(j(S)) \geq 4\left(\sinh\left(\frac{\epsilon_3}{2}\right) - \sinh\left(\frac{\epsilon}{2}\right)\right) \cdot \sqrt{\mathfrak{q}_\phi(j_*[\partial S])},$$

where $j_*[\partial S] \in \partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$. The same holds for any edge $e \in \mathrm{Edg}(\Lambda)$ in place of v above.

Remark 5.6. An elementary computation in hyperbolic geometry yields that $4\sinh(\frac{\epsilon}{2})$ is the Euclidean length of the shortest geodesic on the boundary of a hyperbolic horocusp whose injectivity radius is at most ϵ (realized at points on the boundary). Moreover, the right-hand side of the inequality may be replaced by $\left(1 - 4\sinh\left(\frac{\epsilon}{2}\right)\right) \cdot \sqrt{\mathfrak{q}_\phi(j_*[\partial S])}$, if one takes mutually disjoint maximal horocusps instead of the Margulis horocusps in the definition of $\mathcal{W}_e$. This follows because the length of shortest geodesic on each component of ∂W_e in this

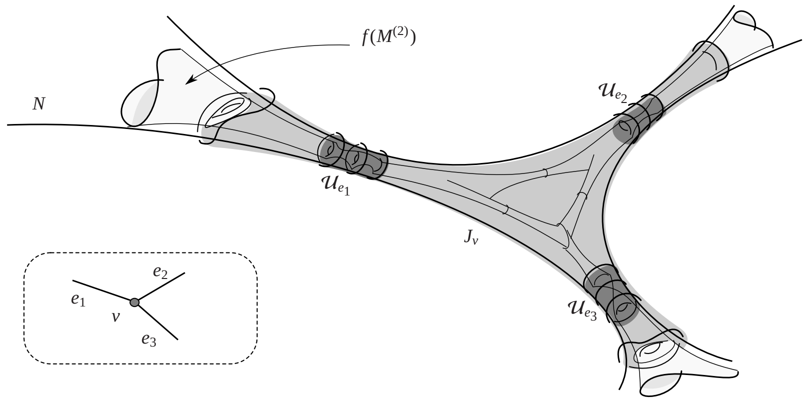

FIGURE 3. A cartoon of N near a piece J_v with three adjacent tori. The neighborhood $\mathcal{W}_v$ of J_v is shaded in light grey. The image of $M^{(2)}$ under f is straight outside the U_e's. In the box is the correponding part of the graph Λ of the geometric decomposition.

case is at least 1, (cf. [Ad]). The author thanks Ian Agol for this interesting improvement, although we shall only use the weaker inequality above to avoid technicalities.

Proof. We only prove the vertex case, and the edge case is similar. Let $v \in \mathrm{Ver}(\Lambda)$ be a vertex. Write $\mathrm{Edg}(v)$ for the edges adjacent to v, and $\widetilde{\mathrm{Edg}}(e)$ for the two ends of an edge e. As Λ is loopless, $e \in \mathrm{Edg}(v)$ has two ends $\delta, \bar{\delta}$, corresponding to the two components of $\mathcal{W}_e \setminus \mathring{\mathcal{U}}_e$ which we write as $\mathcal{W}_\delta, \mathcal{W}_{\bar{\delta}}$ respectively. Suppose $j_*[\partial S] = \sum_{e\in \mathrm{Edg}(v)} \alpha_e$, corresponding to the direct-sum decomposition:

$$H_1(\partial \mathcal{W}_v; \mathbf{R}) \cong \bigoplus_{e\in\mathrm{Edg}(v)} H_1(T_e; \mathbf{R}).$$

It follows from an easy calibration argument that the area (or the horizontal-area) of $j(S) \cap \mathcal{W}_\delta$ is at least $4\,(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2}))\cdot \sqrt{\mathfrak{q}_{J_\delta}(\alpha_e)}$, for any $\delta \in \widetilde{\mathrm{Edg}}(e)$ and any $e \in \mathrm{Edg}(v)$, where $J_\delta \subset \mathcal{J}$ denotes the piece corresponding to the vertex that e is adjacent to on the end δ, (cf. Subsection 3.2 for the definition of $\mathfrak{q}_{J_\delta}$). We have:

$$\begin{aligned}
\mathrm{Area}(j(S)) &\geq \sum_{e\in\mathrm{Edg}(v)} \sum_{\delta\in\widetilde{\mathrm{Edg}}(e)} 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)\cdot \sqrt{\mathfrak{q}_{J_\delta}(\alpha_e)} \\
&\geq 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)\cdot \sqrt{\sum_{e\in\mathrm{Edg}(v)} \sum_{\delta\in\widetilde{\mathrm{Edg}}(e)} \mathfrak{q}_{J_\delta}(\alpha_e)} \\
&= 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)\cdot \sqrt{\sum_{e\in\mathrm{Edg}(v)} \mathfrak{q}_\phi(\alpha_e)} \\
&= 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)\cdot \sqrt{\mathfrak{q}_\phi(j_*[\partial S])},
\end{aligned}$$

as desired. □

The next lemma relates the domination assumption to the geometry of $\mathcal{W}_e$'s and $\mathcal{W}_v$'s. For this purpose, suppose f has already been pulled straight satisfying the conclusion of

Lemma 5.4. For any vertex $v \in \mathrm{Ver}(\Lambda)$, we write $M^{(2)}_{\mathcal{W}_v}$ for $f^{-1}(\mathcal{W}_v) \cap M^{(2)}$, and $M^{(2)}_{\partial\mathcal{W}_v}$ for $f^{-1}(\partial\mathcal{W}_v) \cap M^{(2)}$. For any edge $e \in \mathrm{Ver}(\Lambda)$, we write $M^{(2)}_{\partial\mathcal{W}_e}$, $M^{(2)}_{\partial\mathcal{W}_e}$ with similar meanings.

Lemma 5.7. *If $f : M \to N$ has nonzero degree, then the induced homomorphism:*

$$f|_* : H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R}) \to H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R}),$$

is surjective for any vertex $v \in \mathrm{Ver}(\Lambda)$. The same holds for any edge $e \in \mathrm{Edg}(\Lambda)$ in place of v above.

Proof. We only prove the vertex case, and the edge case is similar. This basically follows from the Poincaré-Lefschetz duality. We first decompose the homomorphism $f|_*$ as:

$$\begin{tikzcd}
H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R}) \arrow[d, "\cong"'] & & \\
H_2(M^{(2)}, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \arrow[r, "i_*"] & H_2(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \arrow[r, "\bar{f}_*"] & H_2(N, N\setminus\mathring{\mathcal{W}}_v; \mathbf{R}) \arrow[d, "\cong"'] \\
& & H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R}),
\end{tikzcd}$$

where the vertical isomorphisms are homology excisions. The homomorphism i_* induced by the inclusion is surjective by the long exact sequence of relative homology:

$$\cdots \longrightarrow H_2(M^{(2)}, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \xrightarrow{i_*} H_2(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \longrightarrow H_2(M, M^{(2)}; \mathbf{R}) \longrightarrow \cdots,$$

where $H_2(M, M^{(2)}; \mathbf{R}) \cong 0$. It suffices to show $\bar{f}_*$ is surjective.

Because $f : M \to N$ has nonzero degree, the commutative diagram:

$$\begin{tikzcd}
H^3(N,\, N\setminus\mathring{\mathcal{W}}_v; \mathbf{R}) \arrow[r, "\bar{f}^*"] \arrow[d] & H^3(M,\, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \arrow[d] \\
H^3(N; \mathbf{R}) \arrow[r, "f^*"] & H^3(M; \mathbf{R}),
\end{tikzcd}$$

implies that $\bar{f}^*$ is injective on the third $\mathbf{R}$-coefficient relative cohomology. Thus,

$$\bar{f}^* : H^*(N,\, N\setminus\mathring{\mathcal{W}}_v; \mathbf{R}) \to H^*(M,\, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}),$$

is injective on all dimensions, following from the commutative diagram:

$$\begin{tikzcd}
H^i(N - N\setminus\mathring{\mathcal{W}}_v; \mathbf{R}) \times H^{3-i}(N,\, N\setminus\mathring{\mathcal{W}}_v; \mathbf{R}) \arrow[r, "\smile"] \arrow[d, "\bar{f}^*"'] & H^3(N,\, N\setminus\mathring{\mathcal{W}}_v; \mathbf{R}) \arrow[d, "\bar{f}^*"'] \\
H^i(M - M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \times H^{3-i}(M,\, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \arrow[r, "\smile"] & H^3(M,\, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}),
\end{tikzcd}$$

where the cup-product pairings are nonsingular and the rightmost vertical homomorphism is injective.

Therefore, $\bar{f}_* : H_*(M,\, M^{(2)}_{N\setminus\mathring{\mathcal{W}}_v}; \mathbf{R}) \to H_*(N,\, N\setminus\mathring{\mathcal{W}}_v; \mathbf{R})$ is indeed surjective on all dimensions, and in particular, on dimension two as desired. □

Observe that every $H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R})$ and every $H_2(M^{(2)}_{\mathcal{W}_e}, M^{(2)}_{\partial\mathcal{W}_e}; \mathbf{R})$ has a spanning set of bounded area:

Lemma 5.8. *For any vertex $v \in \mathrm{Ver}(\Lambda)$, there is an* $\mathbf{R}$*-spanning set of* $H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R})$ *whose elements are represented by relative* $\mathbf{Z}$*-cycles each with area bounded by* $A(2t)$*, where* $A(n) = 27^n(9n^2 + 4n)\pi$ *and* t *is the number of tetrahedra in the triangulation of* M*. The same holds for any edge* $e \in \mathrm{Edg}(\Lambda)$ *in place of* v *above.*

Proof. The argument basically repeats that of [AL, Lemma 3.4], solving integral linear equations associated with presentations of the relative homology modules. The only difference in our situation is that besides the handles and isolated disks described there, $M^{(2)}_{\mathcal{W}_v}$ may contain some 'cornered handles', namely sectors whose boundary contains a 0-simplex of M. To adapt to the previous argument, one may remove an open regular neighborhood $\mathcal{N}$ of $M^{(0)}$ in $M^{(2)}_{\mathcal{W}_v}$, namely, let $\hat{M}^{(2)}_{\mathcal{W}_v} = M^{(2)}_{\mathcal{W}_v} \setminus \mathcal{N}$, and let $\hat{M}^{(2)}_{\partial\mathcal{W}_v} = (M^{(2)}_{\partial\mathcal{W}_v} \cup \bar{\mathcal{N}}) \setminus \mathcal{N}$. Then the argument of [AL, Lemma 3.4] works perfectly for the pair $(\hat{M}^{(2)}_{\mathcal{W}_v}, \hat{M}^{(2)}_{\partial\mathcal{W}_v})$ in place of the pair $(K_V, K_{\partial V})$ there. As $H_2((\hat{M}^{(2)}_{\mathcal{W}_v}, \hat{M}^{(2)}_{\partial\mathcal{W}_v}); \mathbf{R}) \cong H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R})$ via an obvious quotient map $\hat{M}^{(2)}_{\mathcal{W}_v} \to M^{(2)}_{\mathcal{W}_v}$, we obtain an $\mathbf{R}$-spanning set of $H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R})$ represented by relative $\mathbf{Z}$-cycles of area at most $A(2t)$ from that of $H_2((\hat{M}^{(2)}_{\mathcal{W}_v}, \hat{M}^{(2)}_{\partial\mathcal{W}_v}); \mathbf{R})$. □

We need another *a priori* estimation before proving Proposition 5.3:

Lemma 5.9. *For any vertex $v \in \mathrm{Ver}(\Lambda)$, if $\alpha_1, \cdots, \alpha_m$ are elements of $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$ spanning $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$ over $\mathbf{R}$, then for at least one $1 \le k \le m$,*

$$\sqrt{\mathfrak{q}_\phi(\alpha_k)} \ge \mathscr{D}_v(\phi).$$

The same holds for any edge $e \in \mathrm{Edg}(\Lambda)$ *in place of* v *above.*

Proof. This is a Minkowski-type estimation for lattices. Without loss of generality, we may assume that m is minimal, and hence equal to the valence of v. Consider the volume of the parallelogram spanned by the α_i's with respect to the inner product induced by $\mathfrak{q}_\phi$ on $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$, then clearly:

$$\prod_{i=1}^{n_v} \sqrt{\mathfrak{q}_\phi(\alpha_i)} \ge |\det(\alpha_1, \cdots, \alpha_{n_v})| \cdot \sqrt{\Delta(\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z}), \mathfrak{q}_\phi)},$$

where $\det(\alpha_1, \cdots, \alpha_{n_v})$ is the determinant regarding α_i's as column coordinate vectors over a basis of $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$, which is a nonzero integer and hence at least one in absolute value. Thus,

$$\prod_{i=1}^{n_v} \sqrt{\mathfrak{q}_\phi(\alpha_i)} \ \ge \ \sqrt{\Delta(\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z}), \mathfrak{q}_\phi)} \ = \ \mathscr{D}_v(\phi)^{n_v},$$

by the definition of vertex distortion. The lemma follows immediately from this estimation, and the edge case is similar. □

We are now ready to prove Proposition 5.3.

Proof of Proposition 5.3. As the number of vertices and edges of Λ is already bounded in terms of the Kneser-Haken number of M (cf. [BRW, Lemma 4.2]), it suffices to show $\mathscr{D}_v(\phi)$ for any vertex $v \in \mathrm{Ver}(\Lambda)$ and $\mathscr{D}_e(\phi)$ for any edge $e \in \mathrm{Edg}(\Lambda)$ are both bounded in terms of the triangulation number t of M. We only prove the vertex case, and the edge case is similar.

Let $v \in \mathrm{Ver}(\Lambda)$ be a vertex. With the notations of this subsection, we pull straight the nonzero degree map $f : M \to N$ so that it satisfies the conclusion of Lemma 5.4. By Lemma 5.8, there is an $\mathbf{R}$-spanning set $[S_1], \cdots, [S_m]$ of $H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R})$ represented

by relative $\mathbf{Z}$-cycles each with area bounded by $A(2t)$, where $A(n) = 27^n(9n^2+4n)\pi$. From the construction, these relative $\mathbf{Z}$-cycles can be regarded as proper immersions of compact oriented surfaces: $j_i : (S_i, \partial S_i) \to (\mathcal{W}_v, \partial\mathcal{W}_v)$, where $1 \leq i \leq m$. By Lemma 5.5,

$$\begin{aligned}\sqrt{\mathfrak{q}_\phi(j_{i*}[\partial S_i])} &\leq \frac{1}{4}\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)^{-1} \cdot \mathrm{Area}(j_i(S_i)) \\ &\leq \frac{1}{4}\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)^{-1} \cdot A(2t),\end{aligned}$$

for any $1 \leq i \leq m$. Note $j_{i*}[\partial S_i] = \partial_* j_{i*}[S_i] \in \partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$ for any $1 \leq i \leq m$. On the other hand, Lemma 5.7 implies that all the $j_{i*}[\partial S_i]$'s together span $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$ over $\mathbf{R}$, provided that f has nonzero degree. Thus, by Lemma 5.9,

$$\mathscr{D}_v(\phi) \leq \max_{1\leq i\leq m} \sqrt{\mathfrak{q}_\phi(j_{i*}[\partial S_i])} \leq \frac{1}{4}\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)^{-1} \cdot A(2t).$$

As $\epsilon > 0$ can be arbitrarily small, we obtain:

$$\mathscr{D}_v(\phi) \leq \frac{A(2t)}{4\sinh(\frac{\epsilon_3}{2})},$$

where the right-hand side depends only on t as desired. In fact, one can show $\mathscr{D}_v(\phi) \leq A(2t)$ with the stronger estimation as in Remark 5.6. □

Combining Lemma 5.2 and Proposition 5.3, we have proved Proposition 5.1. We close this section by summarizing the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose M is an orientable closed 3-manifold dominating an orientable closed irreducible non-geometric 3-manifold N. As explained in Section 2, there are at most finitely many allowable homeomorphism types of the geometric pieces of N, and the number of geometric pieces in the geometric decomposition of N is bounded in terms of the Kneser-Haken number of M. Thus there are at most finitely many possible preglue graph-of-geometrics so that N is obtained from one of them via a nondegenerate gluing. By Proposition 5.1, the primary distortion $\mathscr{D}(N)$ is bounded in terms of M, so by Proposition 4.1, there are at most finitely many allowable nondegenerate gluings up to equivalence. Hence we conclude that there are at most finitely many allowable homeomorphism types of N, as equivalent gluings yield homeomorphic 3-manifolds. □

6. Domination of bounded degree

In this section, we prove Corollary 1.3. First observe the following simplification:

Lemma 6.1. *If Corollary 1.3 holds under the assumption that the target is a closed orientable Seifert-fibered* 3*-manifold over an orientable base and with nontrivial Euler classes, it holds in general as well.*

Proof. One may first reduce the statement of Corollary 1.3 to the case when the target is irreducible, because any orientable closed 3-manifold 1-dominates any of its connected-sum components in the Kneser-Milnor decomposition, and because the number of connect-sum components in the target is bounded in terms of the Kneser-Haken number of the source. Moreover, Theorem 1.1 reduces the statement to the case of geometric targets, and Theorems 2.1, 2.2 reduce the statement to the case when the targets support one of the geometries $\mathbb{S}^3$, Nil or $\widetilde{\mathrm{SL}}_2$. These are precisely closed orientable Seifert-fibered 3-manifolds with nontrivial Euler classes. To further reduce to the case when the base-orbifold is orientable, we apply the same argument as that of Lemma 5.2, getting rid of essentially embedded Klein-bottles. □

Now it suffices to show the following proposition.

Proposition 6.2. *For any integer $d > 0$, every orientable closed 3-manifold d-dominates at most finitely many orientable-based Seifert-fibered 3-manifolds with nontrivial Euler class.*

Proposition 6.2 is known when d equals one, due to Claude Hayat-Legrand, Shicheng Wang, Heiner Zieschang for the $\mathbb{S}^3$-geometric case ([HWZ]), and due to Shicheng Wang, Qing Zhou for the Nil-geometric and the $\widetilde{\mathrm{SL}}_2$-geometric cases ([WZ]). Their arguments almost work for the general case, but one needs to strengthen a lemma used in both of the papers bounding the size of torsions in the first homology of the targets for general $d > 0$.

Lemma 6.3 (Compare [HWZ, Lemma 3], [WZ, Lemma 3 (1)])**.** *For any integer $d > 0$, if M is an orientable closed 3-manifold d-dominating an orientable closed 3-manifold N, then:*

$$|\mathrm{Tor}\, H_1(N;\mathbf{Z})| \;\leq\; d \cdot |H_1(M;\mathbf{Z}_d)| \cdot |\mathrm{Tor}\, H_1(M;\mathbf{Z})|,$$

where Tor *denotes the torsion submodule, and $|\cdot|$ denotes the cardinality.*

Proof. This follows from an easy algebraic topology argument. Suppose $f : M \to N$ is a map of degree d (after approriately orientate M and N), then the umkehr homomorphism:

$$f_! : H_*(N;\mathbf{Z}) \to H_*(M;\mathbf{Z}),$$

is known as $f_!(\alpha) = [M] \frown f^*(\check{\alpha})$ for $\alpha \in H_*(N;\mathbf{Z})$, where $\check{\alpha} \in H^{3-*}(N;\mathbf{Z})$ denotes the Poincaré dual of α. It is straightforward to check $f_* \circ f_! : H_*(N;\mathbf{Z}) \to H_*(N;\mathbf{Z})$ is the scalar multiplication by d. In particular, $d \cdot \mathrm{Tor}\, H_1(N;\mathbf{Z})$ is surjected by $f_!(\mathrm{Tor}\, H_1(N;\mathbf{Z})) \leq \mathrm{Tor}\, H_1(M;\mathbf{Z})$. On the other hand, from the long exact sequence:

$$\cdots \longrightarrow H_1(N;\mathbf{Z}) \xrightarrow{d} H_1(N;\mathbf{Z}) \longrightarrow H_1(N;\mathbf{Z}_d) \longrightarrow 0,$$

we have $\mathrm{Tor}\, H_1(N;\mathbf{Z}) \,/\, d \cdot \mathrm{Tor}\, H_1(N;\mathbf{Z}) \leq H_1(N;\mathbf{Z}_d)$. Note as $f : M \to N$ has degree d, the image of $H_1(M;\mathbf{Z}_d)$ in $H_1(N;\mathbf{Z}_d)$ has index at most d. This gives our inequality as desired. ◻

Proof of Proposition 6.2. With Lemma 6.3, it follows from the same argument as in [HWZ] and [WZ]. For the reader's reference, we give a brief outline as below. We shall denote an orientable closed Seifert-fibered 3-manifold as $N = \Sigma(g; b_0, \frac{b_1}{a_1}, \cdots, \frac{b_s}{a_s})$, normalized so that $s, g \geq 0$ and b_0 are integers, and that $0 < b_i < a_i$ are coprime integers for $1 \leq i \leq s$. Such a Seifert-fibered 3-manifold fibers over the orientable 2-orbifold $F_g(a_1, \cdots, a_s)$, namely the orientable closed surface of genus g with cone-points of order a_i's, which has Euler characteristic:

$$\chi = 2 - 2g - \sum_{i=1}^{s}(1 - \frac{1}{a_i}),$$

and the Euler class of the fibration (as a rational number) is:

$$e = -b_0 - \sum_{i=1}^{s} \frac{b_i}{a_i},$$

and the torsion size in its first homology is:

$$|\mathrm{Tor}\, H_1(N;\,\mathbf{Z})| = |e| \cdot \prod_{i=1}^{s} a_i,$$

as we have assumed $e \neq 0$.

Suppose M is an orientable closed 3-manifold d-dominating an N as above with $e \neq 0$. Following [HWZ] and [WZ], we consider cases according to the sign of χ, respectively.

When $\chi > 0$, we have $g = 0$ and $s \leq 3$. For $0 \leq s \leq 2$, N is a lens space (possibly the 3-sphere), so there are only finitely many allowable N's up to homeomorphism by [HWZ, Corollary 1], using the linking parings on $\mathrm{Tor}\, H_1(N; \mathbf{Z})$. For $s = 3$, N is either a prism 3-manifold $\Sigma(0; b_0, \frac{1}{2}, \frac{1}{2}, \frac{b_3}{a_3})$, or of one of the types $\Sigma(g; b_0, \frac{1}{2}, \frac{b_2}{3}, \frac{b_3}{3})$, $\Sigma(g; b_0, \frac{1}{2}, \frac{b_2}{3}, \frac{b_3}{4})$, or $\Sigma(g; b_0, \frac{1}{2}, \frac{b_2}{3}, \frac{b_3}{5})$. For the latter three types, b_2, b_3 are automatically bounded by their denominators, and one can bound b_0 by Lemma 6.3 and by the formulae above. For the prism case, N admits a $\mathbf{Z}_2$-action whose quotient is a lens space L with $\pi_1(L)$ isomorphic to either $\mathbf{Z}_{2\cdot|a_3b_0+b_3-a_3|}$ or $\mathbf{Z}_{|a_3b_0+b_3-a_3|}$, according to a_3 being odd or even respectively, so $|a_3b_0 + b_3 - a_3|$ are bounded by the lens space case as M $2d$-dominates L. Because the torsion-size comparison bounds $|a_3b_0 + b_3 + a_3|$, and because $0 < b_3 < a_3$, we have upperbounds on $|b_0|, a_3, b_3$ in this case as well. Thus there are at most finitely many homeomorphically distinct N's with $\chi > 0$.

When $\chi = 0$, there are only finitely many allowable values of s, g and a_i's by the formula of χ. For each possibility, there are only finitely allowable values of b_i's because $0 < b_i < a_i$ for $1 \leq i \leq s$, and because b_0 can be bounded by the torsion-size comparison. Thus there are at most finitely many homeomorphically distinct N's with $\chi = 0$.

When $\chi < 0$, we have $\chi \leq -\frac{1}{42}$ realized when the base-orbifold is the turnover $F_0(2, 3, 7)$. Using the Seifert volume introduced in [BG], we have $\mathrm{SV}(M) \geq d \cdot \mathrm{SV}(N) = |e|^{-1}\chi^2 d$, (cf. [WZ, Lemmas 3 (2), 4 (3)]). Thus $|e|$ is bounded from zero in terms of M. By Lemma 6.3 and the torsion-size formula above, we can bound the s and the values of a_i's. This in turn yields an upperbound of $|b_0|$ applying the torsion-size comparison again. Thus there are at most finitely many homeomorphically distinct N's with $\chi < 0$ as well. This completes the proof of Proposition 6.2. □

Proof of Corollary 1.3. It follows immediately from Lemma 6.1 and Proposition 6.2. □

7. Conclusion

We would like to view Theorem 1.1 and Corollary 1.3 as results about maps between 3-manifolds. A couple of further questions are suggested.

Question 7.1. Which groups surject at most finitely many isomorphically distinct fundamental groups of aspherical 3-manifolds?

Such group are sometimes said to be *tiny*. For example, virtually solvable groups are tiny groups. It seems that our techniques imply that finitely generated groups with vanishing first Betti number are also tiny.

Let M be an orientable closed 3-manfold. For any integer $d > 0$, denote the number of homeomorphically distinct 3-manifolds dominated by M of degree at most d as $\tau_M(d)$. By Corollary 1.3, $\tau_M(d)$ is a positive finite integer.

Question 7.2. Is it possible to decide τ_M for $d > 0$ sufficiently large?

It is an old problem to decide whether a given orientable closed 3-manifold M dominates (or d-dominates) another given N ([Wa, Question 1.1]). Question 7.2 is supposably a much weaker version of that problem.

Question 7.3. For an orientable closed nontrivial graph manifold N, is there an explicit bound of its Seifert volume $\mathrm{SV}(N)$ in terms of its graph $\Lambda(N)$ and its average distortions $\mathscr{D}_v(N)$'s and $\mathscr{D}_e(N)$'s?

This question is motivated by the fact that the Seifert volume (cf. [BG]) also reflects the complexity of gluings. More generally, we wonder if there is a more insightful definition of the 'global distortion' of an orientable closed 3-manifold N, other than the primary distortion $\mathscr{D}(N)$ introduced in this paper.

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